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Deformation quantization using groupoids. The case of toric manifolds

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Abstract

In the framework of C*-algebraic deformation quantization we propose a notion of a deformation groupoid which could apply to known examples such as Connes's tangent groupoid of a manifold, its generalization by Landsman and Ramazan, Rieffel's noncommutative torus, and even Landi's noncommutative 4-sphere. We construct such a groupoid for a wide class of $\mathbb{T}^n$-spaces, that generalizes the one given for $\mathbb{C}^n$ by Bellissard and Vittot. In particular, using the geometric properties of the moment map discovered in the 1980's by Atiyah, Delzant, Guillemin and Sternberg, it provides a C*-algebraic deformation quantization for all toric manifolds, including the 2-sphere and all complex projective spaces.

0. Introduction

Quantization is going from classical mechanics to quantum mechanics. In the classical Hamiltonian picture, a physical system is described by its phase space, a Poisson manifold M, with Poisson bivector denoted by Ω, and an associated Poisson bracket of two observables $\mathtt{f}, \mathtt{f}' \in \mathrm{C}^\infty(M)$:

$$\{\mathtt{f}, \mathtt{f}'\}_\Omega \stackrel{def}{=} \left\langle d\mathtt{f} \otimes d\mathtt{f}' \mid \Omega \right\rangle.$$

On the other hand, a quantum system is described, from Heisenberg's point of view, by replacing the classical algebra $\mathrm{C}^\infty(M)$ of observables by an algebra of operators on a Hilbert space. Moreover the Poisson bracket $\{\mathtt{f}, \mathtt{f}'\}_\Omega$ has to be replaced by $\frac{1}{\imath\hbar}[f, f']$, where f and f' are the quantum analogs of $\mathtt{f}$ and $\mathtt{f}'$, $[f, f']$ is their commutator, and $\hbar$ is Planck's physical constant. Dirac postulated in the 1930's that the correspondence from renormalized commutators to Poisson bracket is given by a limit process when Planck's constant goes to zero.

In the 1970's the notion of *deformation quantization* was introduced in [3] as an attempt to give a precise definition to Dirac's principle. One of the ideas was to use a field of algebras $(\mathcal{A}_\hbar)$ parametrized by a real number $\hbar$, such that, for $\hbar = 0$, the algebra $\mathcal{A}_0$ is the commutative algebra of classical observables, and for $\hbar \neq 0$, $\mathcal{A}_\hbar$ is the noncommutative algebra of quantum observables. In the early 1990's Rieffel proposed in [21,22] a topological definition of the limit process making use of the previously known notion of the continuous field of C*-algebras [10]. The following is a slight generalization of Rieffel's own, but adapted to groupoid C*-algebras.

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Definition 0.1. Let $(\mathcal{A}_\hbar)_{\hbar \in X}$ be a continuous field of C*-algebras parametrized by a locally compact subset X of $\mathbb{R}$ containing 0 as a limit point; we denote by $\mathcal{A} = \bigcup_{\hbar \in X} \mathcal{A}_\hbar$ the associated topological bundle over X. Let $\mathcal{Q}$ be a sub-*-algebra of the C*-algebra $C_0(X, \mathcal{A})$ of continuous *sections* of $\mathcal{A}$. We say that $((\mathcal{A}_\hbar)_{\hbar \in X}, \mathcal{Q})$ is a *deformation* if:

- The space $\mathcal{Q}_0 = \{f_0 \in \mathcal{A}_0 \mid f \in \mathcal{Q}\}$ is dense in $\mathcal{A}_0$.
- There is a map $\mathcal{Q}_0 \times \mathcal{Q}_0 \longrightarrow \mathcal{Q}_0$, denoted by

  $$(a, b) \longmapsto \{a, b\}_0,$$

  such that, for every $f, f' \in \mathcal{Q}$, the bracket $\{f_0, f'_0\}_0$ is the continuous extension at zero of the continuous section $\hbar \longmapsto \frac{1}{\imath\hbar}[f_\hbar, f'_\hbar]$ defined on $X - \{0\}$ *i.e.*:

  $$\{f_0, f'_0\}_0 = \lim_{\substack{\hbar \to 0 \\ \hbar \neq 0}} \frac{1}{\imath\hbar}[f_\hbar, f'_\hbar].$$

Thus, $\mathcal{A}_0$ is a commutative C*-algebra and $\mathcal{Q}_0$ is a sub-*-algebra, and, moreover, $\mathcal{Q}_0$ has a structure of Poisson algebra for the bracket $\{.,.\}_0$.

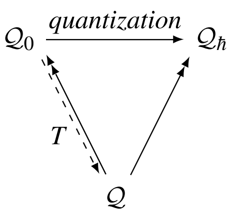
Definition 0.2. Let (M, Ω) be a Poisson manifold. We denote as $C_0(M)$ (resp. $C_0^\infty(M)$) the algebra of functions on M vanishing at infinity (resp. the algebra of smooth functions on M vanishing at infinity). A *deformation of* M is a deformation $((\mathcal{A}_\hbar)_{\hbar \in X}, \mathcal{Q})$ endowed with an isomorphism of C*-algebras:

$$C_0(M) \xrightarrow{\mathcal{I}} \mathcal{A}_0,$$

such that:

- $\mathcal{Q}_0 \subset \mathcal{I}(C_0^\infty(M))$;
- for each $f, f' \in \mathcal{Q}$, $\{f_0, f'_0\}_0 = \mathcal{I}\left(\{\mathcal{I}^{-1} f_0, \mathcal{I}^{-1} f'_0\}_\Omega\right)$.

Then, each linear *-preserving section T of the canonical projection of $\mathcal{Q}$ onto $\mathcal{Q}_0$ gives rise to a quantization:



Many usual C*-algebras can be described as the C*-algebra $C^*(G)$ of a groupoid G, using the construction given by Jean Renault in [18] that generalizes the construction of the (full and reduced) C*-algebra(s) of a group or of a group action. Note that a bundle of groupoids is also a groupoid. On the topological level Ramazan [17,14] established the following property:

Theorem 0.3. *Let $(G_\hbar)_{\hbar \in X}$ be a field of groupoids, and $G = \bigcup_{\hbar \in X} G_\hbar$ be the corresponding bundle, which we assume to be locally compact, Hausdorff, separable, endowed with a continuous Haar system, amenable (cf.* [1]*), and such that the bundle map $G \xrightarrow{p} X$ is open. Then, the field $\left(C^*(G_\hbar)\right)_{\hbar \in X}$ possesses a continuous field structure such that the algebra of continuous sections is*

$$C_0\left(X, \bigcup_{\hbar \in X} C^*(G_\hbar)\right) = C^*(G).$$

Definition 0.4. Let G be a locally compact, Hausdorff, separable bundle of groupoids over a locally compact subset X of $\mathbb{R}$, endowed with a continuous Haar system, amenable, and such that the bundle map $G \xrightarrow{p} X$ is open. We say that G is a *deformation groupoid* if there exists a sub-*-algebra $\mathcal{Q}$ of the algebra of continuous sections of the field

$\left(\mathrm{C}^*(G_\hbar)\right)_{\hbar\in X}$ such that $\left(\left(\mathrm{C}^*(G_\hbar)\right)_{\hbar\in X}, \mathcal{Q}\right)$ is a deformation. We say that G is a *deformation groupoid of a Poisson manifold* M if there exists such $\mathcal{Q}$ and an isomorphism $\mathcal{I}$ such that $\left(\left(\mathrm{C}^*(G_\hbar)\right)_{\hbar\in X}, \mathcal{Q}, \mathcal{I}\right)$ is a deformation of M.

If G is as in Definition 0.4, then G is a deformation groupoid of (M, Ω) if and only if:

1. There exists an isomorphism $\mathrm{C}_0(M) \xrightarrow[\sim]{\mathcal{I}} \mathrm{C}^*(G_0)$.
2. There exists a sub-*-algebra $\mathcal{Q}$ of $\mathrm{C}^*(G)$ such that:
   (a) $\mathcal{Q}_0$ is dense in $\mathrm{C}^*(G_0)$,
   (b) $\mathcal{I}^{-1}(\mathcal{Q}_0) \subset \mathrm{C}_0^\infty(M)$,
   (c) $\forall f, f' \in \mathcal{Q},\ \lim_{\substack{\hbar\to 0\\ \hbar\neq 0}} \frac{1}{\imath\hbar}[f_\hbar, f'_\hbar] = \mathcal{I}\left(\{\mathcal{I}^{-1} f_0, \mathcal{I}^{-1} f'_0\}_\Omega\right)$.

We will need the following result:

Proposition 0.5. *Let M be a manifold, without a priori Poisson structure. Suppose there exists G as in Definition* 0.4 *such that G satisfies the conditions* 1, 2(a), 2(b) *above, and the following condition:*

2(c') *for every $f, f' \in \mathcal{Q}$ the section $\hbar \longrightarrow \frac{1}{\imath\hbar}[f_\hbar, f'_\hbar]$ has a unique continuous extension at $\hbar = 0$ in $\mathcal{Q}_0$, which only depends on the values f_0 and f'_0.*

Then, M admits a Poisson bracket such that G is a deformation groupoid of M.

The first *example* of a deformation groupoid was given by Connes in [8] (see also [23]): the tangent groupoid of a manifold N is a deformation groupoid of the cotangent bundle T^*N endowed with its canonical symplectic structure. This was generalized by Landsman and Ramazan [14,17,13] to integrable Lie–Poisson manifolds (*i.e.* manifolds which are the dual of an integrable Lie algebroid, endowed with the canonical Poisson structure described in [11]). In 1990 Bellissard and Vittot [5] constructed a deformation groupoid of $\mathbb{C}^n$ with its canonical symplectic structure which was not a Lie groupoid. In the present paper, we generalize this construction to other $\mathbb{T}^n$-spaces M (where $\mathbb{T}^n$ denotes the n-dimensional torus). The construction applies to the 2-sphere and, using Delzant's results [9], to all toric manifolds, including all complex projective spaces endowed with their canonical Kähler structure.

The strategy is based on two remarks. First, the existence of an isomorphism $\mathrm{C}_0(M) \xrightarrow[\sim]{\mathcal{I}} \mathrm{C}^*(G_0)$ (condition 1 above) implies by [19, lemme 1.3 p. 7] that the "classical" groupoid G_0 must be a bundle of commutative groups.[1] Then, we note that the Fourier–Gelfand transform $\mathrm{C}(\mathbb{T}^n) \simeq \mathrm{C}^*(\mathbb{Z}^n)$ gives such an isomorphism when $M = \mathbb{T}^n$ and $G = \mathbb{Z}^n$.

The paper is organized as follows:

Section 1 Given a $\mathbb{T}^n$-space M, we construct an isomorphism $\mathrm{C}(M) \simeq \mathrm{C}^*(G_0)$ for a suitable bundle of commutative groups G_0, under the assumption that the projection $M \twoheadrightarrow M/\mathbb{T}^n$ has a continuous section (Theorem 1.3).

Section 2 Using a second action on $M/\mathbb{T}^n$, we then construct a deformation groupoid bundle G over $\mathbb{R}$ such that the fiber at 0 of G is G_0. The sub-*-algebra $\mathcal{Q}$ considered consists of restrictions of C_c^∞ functions on a Lie groupoid $\widetilde{G}$ containing G as subgroupoid (Theorem 2.2).

Section 3 We compute the Poisson structure corresponding to this deformation structure on M (Theorem 3.2).

Section 4 We check that this construction can be applied to toric manifolds, and that the Poisson structure so obtained is the original symplectic structure of the toric manifold (Theorem 4.2).

Another application of this construction is to give a groupoid description of Rieffel's multidimensional noncommutative tori [20] and of Connes and Landi's noncommutative 4-sphere [7]. The details of these examples can be found in [6, pages 64–68].

[1] Note that a bundle of commutative groups is always amenable. Replacing globally continuous fields of C*-algebras by fields only continuous at zero, one could construct nonglobally amenable deformation groupoids.

1. Fourier–Gelfand isomorphisms for a $\mathbb{T}^n$-space

For G a groupoid, we denote by $G^{(0)}$ its unit space, and by $G \underset{s}{\overset{r}{\rightrightarrows}} G^{(0)}$ its range and source maps. The following definition appears in [18]:

Definition 1.1. A locally compact, Hausdorff, separable groupoid G is *étale* when the sets:

$$G^y = \{g \in G \mid r(g) = y\}, \quad y \in G^{(0)} \quad \text{(fibers of the range map)}$$

are discrete, and the counting measure is a continuous Haar system.

Étale groupoids have the following well-known properties:

- An open subgroupoid of an étale groupoid is itself étale.
- The maps r, s are open [18]. And the converse is true if G is a bundle of commutative groups: such a topological bundle is étale if and only if the projection $G \xrightarrow{r=s} G^{(0)}$ is open (cf. [19]).

Let M be a Hausdorff, locally compact, separable space endowed with an action α of $\mathbb{T}^n$ denoted by $(s, x) \in \mathbb{T}^n \times M \longmapsto \alpha_s(x) \in M$. Let us introduce some notation:

- $\Delta = M/\mathbb{T}^n$ denotes the quotient space, which is Hausdorff, separable and locally compact, and let $M \overset{J}{\twoheadrightarrow} \Delta$ be the canonical projection;
- for $y \in \Delta$, $\mathbb{T}^n_y$ is the common isotropy subgroup of all points $x \in M$ such that $J(x) = y$, *i.e.* $\mathbb{T}^n_y = \{s \in \mathbb{T}^n \mid \alpha_s(x) = x\}$;
- $\mathbb{T}^n/\mathbb{T}^n_y$ is the quotient group and $\widehat{\mathbb{T}^n/\mathbb{T}^n_y}$ is its Pontryagin dual. Since $\mathbb{T}^n_y$ is closed in $\mathbb{T}^n$, the dual map of $\mathbb{T}^n \twoheadrightarrow \mathbb{T}^n/\mathbb{T}^n_y$ is injective:
$$\widehat{\mathbb{T}^n/\mathbb{T}^n_y} \hookrightarrow \widehat{\mathbb{T}^n} = \mathbb{Z}^n;$$
- for $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and $s = (s_1, \ldots, s_n) \in \mathbb{T}^n$, we denote the duality bracket
$$\langle k, s\rangle = s_1^{k_1} \cdots s_n^{k_n};$$
- for $k \in \mathbb{Z}^n$, we let $\Delta(k) = \{y \in \Delta \mid \forall s \in \mathbb{T}^n_y, \langle k, s\rangle = 1\}$; then, $k \in \widehat{\mathbb{T}^n/\mathbb{T}^n_y}$ if and only if $y \in \Delta(k)$, and we also let $\Delta(\infty) = \{y \in \Delta \mid \mathbb{T}^n_y = \{1\}\}$.

Finally we construct a bundle of commutative groups G_0 by setting

$$G_0 = \bigcup_{y\in\Delta} \widehat{\mathbb{T}^n/\mathbb{T}^n_y} = \bigcup_{k\in\mathbb{Z}^n} \Delta(k) \times \{k\} \subset \Delta \times \mathbb{Z}^n,$$

equipped with the topology induced by $\Delta \times \mathbb{Z}^n$.

Proposition 1.2. *If there exists a continuous section σ of $J\colon M \underset{J}{\overset{\sigma}{\rightleftarrows}} \Delta$, then $G_0 \twoheadrightarrow \Delta$ is an étale bundle of commutative groups.*

Proof. Since the trivial bundle of groups $\Delta \times \mathbb{Z}^n$ is clearly étale, it suffices to prove that G_0 is open. So it suffices to prove that the $\Delta(k)$ are open in Δ, *i.e.* that their complements $\Delta(k)^c$ are closed in Δ.

First we give a characterization of elements in $\Delta(k)^c$. For each $y \in \Delta$, define a subgroup H_y of $\mathbb{T}$ by $H_y = \{\langle k, s\rangle \mid s \in \mathbb{T}^n_y\}$. We have

$$y \in \Delta(k)^c \Leftrightarrow k \notin \widehat{\mathbb{T}^n/\mathbb{T}^n_y} \Leftrightarrow H_y \neq \{1\}.$$

Moreover if H is a subgroup of $\mathbb{T}$, we have

$$(\forall t \in H, |t - 1| < \sqrt{2}) \Leftrightarrow H = \{1\},$$

which is a particular case of the fact that Lie groups do not have small nontrivial subgroups [16]. Thus,

$$y \in \Delta(k)^c \Leftrightarrow (\exists s \in \mathbb{T}^n_y, |\langle k, s\rangle - 1| \geq \sqrt{2}).$$

Let us now prove that for any convergent sequence $(y_i)_i$ in $\Delta(k)^c$, the limit y is also in $\Delta(k)^c$. Since $y_i \in \Delta(k)^c$, for each i there exists a $s_i \in \mathbb{T}^n_{y_i}$ such that $|\langle k, s_i\rangle - 1| \geq \sqrt{2}$. Let $(s_{i_j})_j$ be a subsequence of $(s_i)_i$ in $\mathbb{T}^n$ converging to some s. Then, $|\langle k, s\rangle - 1| \geq \sqrt{2}$. Moreover, since $s_i \in \mathbb{T}^n_{y_i}$, we have $\alpha_{s_{i_j}}(\sigma(y_{i_j})) = \sigma(y_{i_j})$, for every j. In the limit, we get $\alpha_s(\sigma(y)) = \sigma(y)$, and hence $s \in \mathbb{T}^n_y$. Thus, we have proved

$$\exists s \in \mathbb{T}^n_y, \quad |\langle k, s\rangle - 1| \geq \sqrt{2},$$

which means that $\Delta(k)^c$ is closed in Δ. ■

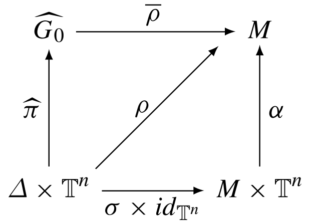
As a corollary, we note that the bundle map $G_0 \xrightarrow{p} \Delta$ is open.

Such continuous sections σ may not exist; in the sequel we suppose that they do, and we introduce more notations which depend implicitly on the choice of σ.

- We define the map $\Delta \times \mathbb{T}^n \xrightarrow{\rho} M$ by $\rho(y, s) = \alpha_s(\sigma(y))$.
- The "dual" bundle of G_0 is

  $$\widehat{G_0} = \bigcup_{y\in\Delta} \mathbb{T}^n/\mathbb{T}^n_y,$$

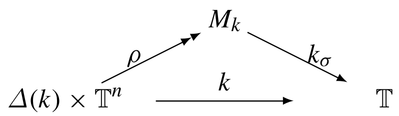
  endowed with the finest topology such that the canonical projection $\Delta \times \mathbb{T}^n \xrightarrow{\widehat{\pi}} \widehat{G_0}$ is continuous.
- Since $\rho(y, s) = \rho(y', s')$ holds if and only if $y = y'$ and $s = s' \mod \mathbb{T}^n_y$, there is a quotient map $\overline{\rho}$ which is one-to-one and onto and that makes the following diagram commutative.



- For every $k \in \mathbb{Z}^n \cup \{\infty\}$, we let M_k be the following subset of M:

  $$M_k = \rho(\Delta(k) \times \mathbb{T}^n) = J^{-1}(\Delta(k)) = \{\alpha_s(\sigma(y)) \mid s \in \mathbb{T}^n,\ y \in \Delta(k)\}.$$

  In particular, note that $M_\infty = J^{-1}(\Delta(\infty))$ is the maximal stable subset of M on which the action of $\mathbb{T}^n$ is free.
- For every $k \in \mathbb{Z}^n$, we define a function k_σ that makes the following diagram commutative:



(*i.e.* $k_\sigma(\rho(y, s)) = \langle k, s\rangle$)

The main result of this section is the following.

Theorem 1.3. *For any continuous section σ of J and any $k \in \mathbb{Z}^n$, the set M_k is open in M, and the function k_σ is continuous. Moreover there is an isomorphism of* C^**-algebras* $C_0(M) \xrightarrow[\sim]{\mathcal{I}} C^*(G_0)$*, such that, for any $f \in C_c(G_0)$ and any $x \in M$,*

$$(\mathcal{I}^{-1} f)(x) = \sum_{k\in\widehat{\mathbb{T}^n/\mathbb{T}^n_y}} f(J(x), k)k_\sigma(x), \quad \textit{where } y = J(x).$$

We decompose the proof into several lemmas.

Lemma 1.4. *The map $\overline{\rho}$ is a homeomorphism.*

Proof. Since the group $\mathbb{T}^n$ is compact, its action α on M is a proper map, and σ is proper too, since, for every compact subset K of M, the set $\sigma^{-1}(K) = J(K)$ is compact. Thus, $\overline{\rho} \circ \widehat{\pi} = \alpha \circ (\sigma \times id_{\mathbb{T}^n})$ is a closed and continuous map. Hence, $\overline{\rho}$ itself is closed, since $\widehat{\pi}$ is continuous and surjective. Since $\overline{\rho}$ is continuous, by the choice of topology of $\widehat{G_0}$, $\overline{\rho}$ is a homeomorphism. ■

We obtain, as immediate corollaries of this lemma, that:

- the bundle map $\widehat{G_0} \xrightarrow{\widehat{p}=J\circ\overline{\rho}} \Delta$ is continuous and proper;
- the bundle duality bracket $\widehat{G_0} \underset{\Delta}{*} G_0 \longrightarrow \mathbb{T}$ is continuous, where $\widehat{G_0} \underset{\Delta}{*} G_0$ denotes the fibered product.

For any function $f \in \mathrm{C_c}(G_0)$, let us define its Fourier transform as the function

$$\begin{array}{ccc} \widehat{G_0} & \overset{\mathcal{F}f}{\to} & \mathbb{C} \\ (y, s \mod \mathbb{T}^n_y) & \mapsto & \displaystyle\sum_{k\in\widehat{\mathbb{T}^n/\mathbb{T}^n_y}} f(y,k)\,\langle k, s\rangle. \end{array}$$

Then, for any $y \in \Delta$, we have $(\mathcal{F}f)_y = (\mathcal{F}_y)(f_y)$, where f_y denotes the restriction of f to $\widehat{\mathbb{T}^n/\mathbb{T}^n_y}$ and $\mathrm{C}^*\left(\widehat{\mathbb{T}^n/\mathbb{T}^n_y}\right) \xrightarrow{\mathcal{F}_y} \mathrm{C}\left(\mathbb{T}^n/\mathbb{T}^n_y\right)$ is the classical Fourier–Gelfand transform for commutative groups.

Lemma 1.5. *The Fourier transform induces an isomorphism of* C^**-algebras*

$$\mathrm{C}^*(G_0) \xrightarrow{\mathcal{F}} \mathrm{C}_0(\widehat{G_0}).$$

Proof. Since $\widehat{G_0}$ is homeomorphic to M, the space $\widehat{G_0} \times G_0$ is normal, and $\widehat{G_0} \underset{\Delta}{*} G_0$ is a closed subset. For any fixed $f \in \mathrm{C_c}(G_0)$, the continuous map $(y, s, k) \longmapsto f(y,k)\langle k, s\rangle$ defined on $\widehat{G_0} \underset{\Delta}{*} G_0$ admits a continuous extension K on $\widehat{G_0} \times G_0$ (by Uhrysohn's Theorem). Then, using [4, chap X, Section 3, no. 4, th. 3] and the fact that G_0 is étale, and hence the counting measure is a *continuous* Haar system, we obtain the continuity of $\mathcal{F}f$, since

$$(\mathcal{F}f)(y, s \bmod \mathbb{T}^n_y) = \sum_{k\in\widehat{\mathbb{T}^n/\mathbb{T}^n_y}} K((y,s),(y,k)).$$

Moreover, the inequality

$$\left|\mathcal{F}f(y, s \bmod \mathbb{T}^n_y)\right| \le \sum_{k\in\widehat{\mathbb{T}^n/\mathbb{T}^n_y}} |f(y,k)|\,,$$

implies that the support of $\mathcal{F}f$ is contained in the set $\widehat{p}^{-1}(p(\mathrm{Supp}\, f))$ which is compact, since $\widehat{p}$ is proper.

We consider $\mathrm{C_c}(G_0)$ as a dense sub-*-algebra of $\mathrm{C}^*(G_0)$, and $\mathrm{C_c}(\widehat{G_0})$ as a sub-*-algebra of $\mathrm{C}_0(\widehat{G_0})$, also dense for the sup norm. An easy calculation shows that the map $\mathrm{C_c}(G_0) \xrightarrow{\mathcal{F}} \mathrm{C_c}(\widehat{G_0})$ is a morphism of commutative *-algebras.

Let us prove now that $\mathcal{F}$ is isometric. Since the bundle map $G_0 \xrightarrow{p} \Delta$ is open, we get a continuous field of C^*-algebras $\left(\mathrm{C}^*(\widehat{\mathbb{T}^n/\mathbb{T}^n_y})\right)_{y\in\Delta}$, whose C^*-algebra of continuous sections is $\mathrm{C}^*(G_0)$; then, for every $f \in \mathrm{C_c}(G_0)$, we get

$$\|f\|_{\mathrm{C}^*(G_0)} = \sup_{y\in\Delta} \|f_y\|_{\mathrm{C}^*(\widehat{\mathbb{T}^n/\mathbb{T}^n_y})}.$$

Since the Fourier transform $\mathcal{F}_y$ is isometric for any $y \in \Delta$, we get

$$\|f_y\|_{\mathrm{C}^*(\widehat{\mathbb{T}^n/\mathbb{T}^n_y})} = \|\mathcal{F}_y(f_y)\|_{\mathrm{C}\left(\mathbb{T}^n/\mathbb{T}^n_y\right)}.$$

Hence,

$$\|f\|_{\mathrm{C}^*(G_0)} = \sup_{y\in\Delta} \|\mathcal{F}_y(f_y)\|_{\mathrm{C}\left(\mathbb{T}^n/\mathbb{T}^n_y\right)} = \|\mathcal{F}f\|_{\mathrm{C}_0(\widehat{G_0})}.$$

Hence, by completion, $\mathcal{F}$ has an isometric extension $\mathrm{C}^*(G_0) \xrightarrow{\mathcal{F}} \mathrm{C}_0(\widehat{G_0})$. Moreover this extension is surjective, and, since $\mathcal{F}(\mathrm{C}^*(G_0))$ is a closed and dense sub-*-algebra, the extension is an isomorphism by the Stone–Weierstrass Theorem.[2] ■

Now we give the proof of the Theorem 1.3.

Proof. The sets $M_k = J^{-1}(\Delta(k))$ are open since J is continuous and the $\Delta(k)$ are open. Moreover since, for all $s \in \mathbb{T}^n$ and $y \in \Delta(k)$,

$$k_\sigma(\rho(y, s)) = \langle k, s\rangle,$$

we have $k_\sigma = k \circ \overline{\rho}^{-1}$, and hence k_σ is continuous.

The isomorphism $\mathcal{I}$ is defined as the map that makes the following diagram commute.

$$\begin{array}{ccc} \mathrm{C}_0(M) & \xrightarrow{\mathcal{I}} & \mathrm{C}^*(G_0) \\ & {\scriptstyle \overline{\rho}^*}\searrow \quad \swarrow{\scriptstyle \mathcal{F}} & \\ & \mathrm{C}_0(\widehat{G_0}) & \end{array} \qquad \left(i.e.\ \mathcal{I}^{-1} f = (\mathcal{F} f) \circ \overline{\rho}^{-1}\right)$$

■

2. Deformation groupoid of a $\mathbb{T}^n$-space

Assume that the $\mathbb{T}^n$-space M is in fact a manifold, and that the quotient space Δ is a locally closed subset of a Hausdorff manifold N endowed with a smooth action β of $\mathbb{R}^n$:

$$\underset{\circlearrowleft_\alpha}{M} \xrightarrow{\quad J \quad} \Delta \subset \underset{\circlearrowleft_\beta}{N}.$$

Then, one can construct a deformation groupoid of M (for some Poisson structure) in the following way: for such a given action β, there is a right action of $\mathbb{Z}^n$ on $\mathbb{R} \times N$ defined for every $\hbar \in \mathbb{R}$, $y \in N$ and $k \in \mathbb{Z}^n$ by

$$(\hbar, y) \cdot k = (\hbar, \beta_{\hbar k}(y)).$$

Let us denote by

$$\widetilde{G} = (\mathbb{R} \times N) \rtimes \mathbb{Z}^n$$

the cross-product groupoid (cf. [18]); then, $\widetilde{G}$ is an étale Lie bundle of groupoids parametrized by $\hbar \in \mathbb{R}$. We let

$$G = \{(\hbar, y, k) \in \mathbb{R} \times N \times \mathbb{Z}^n \mid y, \beta_{\hbar k}(y) \in \Delta(k) \text{ and } \mathbb{T}^n_y = \mathbb{T}^n_{\beta_{\hbar k}(y)}\},$$

where the terms $\Delta(k)$ and $\mathbb{T}^n_y$ are defined in the previous section. Then, the fiber $\{(\hbar, y, k) \in G \mid \hbar = 0\}$ is exactly the groupoid G_0 defined in the previous section.

Proposition 2.1. *Let M be a Hausdorff separable manifold with an action of $\mathbb{T}^n$ such that the quotient space Δ is a locally closed subset of a Hausdorff manifold N, which is endowed with a smooth action β of $\mathbb{R}^n$. Then, the groupoid G previously defined is a subgroupoid of $\widetilde{G}$, with space of units $G^{(0)} = \mathbb{R} \times \Delta$.*

Moreover, setting $G_k = G \cap (\mathbb{R} \times N \times \{k\})$, the groupoid G is étale if and only if, for all $k \in \mathbb{Z}^n$, the sets

$$s(G_k) = \{(\hbar, y) \mid y, \beta_{\hbar k}(y) \in \Delta(k) \text{ and } \mathbb{T}^n_y = \mathbb{T}^n_{\beta_{\hbar k}(y)}\}$$

are open in $\mathbb{R} \times \Delta$; then, G is amenable and the projection $G \twoheadrightarrow \mathbb{R}$ is open.

2 Because, for every distinct $(y, s \bmod \mathbb{T}^n_y)$ and $(y', s' \bmod \mathbb{T}^n_{y'})$ in $\widehat{G_0}$, it is easy to construct an $f \in \mathrm{C_c}(G_0)$ such that $(\mathcal{F} f)(y, s \bmod \mathbb{T}^n_y) \neq (\mathcal{F} f)(y', s' \bmod \mathbb{T}^n_{y'})$.

Proof. For every groupoid G with space of units $G^{(0)}$, and for any subset U of $G^{(0)}$, the set $r^{-1}(U) \cap s^{-1}(U)$ is a subgroupoid of G, called the *restriction of* G *to* U, denoted as $G|_U$, whose space of units is U.

For every closed subgroup H of $\mathbb{T}^n$, $\widehat{\mathbb{T}^n/H}$ is a subgroup of $\mathbb{Z}^n$, and hence $(\mathbb{R} \times N) \rtimes \widehat{\mathbb{T}^n/H}$ is a subgroupoid of $\widetilde{G}$. We let

$$\Delta_H = \{y \in \Delta \mid \mathbb{T}^n_y = H\}.$$

Then, we get a partition $\Delta = \bigcup_H \Delta_H$ and we can form the restriction $\left((\mathbb{R} \times N) \rtimes \widehat{\mathbb{T}^n/H}\right)\Big|_{\mathbb{R}\times\Delta_H}$. Since, for every $k \in \mathbb{Z}^n$, we have the partition

$$\Delta(k) = \bigcup_{H \text{ s.t. } k \in \widehat{\mathbb{T}^n/H}} \Delta_H,$$

it is easy to verify that G is the disjoint union $G = \bigcup_H \left((\mathbb{R} \times N) \rtimes \widehat{\mathbb{T}^n/H}\right)\Big|_{\mathbb{R}\times\Delta_H}$, and hence a subgroupoid of $\widetilde{G}$.

A locally compact, Hausdorff, separable groupoid with open source and open range is étale if and only if it admits a covering by open bisections. Hence a subgroupoid G of an étale groupoid $\widetilde{G}$ is itself étale if and only if $G^{(0)}$ is locally closed in $\widetilde{G}^{(0)}$, and the images by the source map s (or by the range r) of open bisections covering G are open in $G^{(0)}$. In our case, the G_k are bisections covering G.

The bundle map $G \xrightarrow{p} \mathbb{R}$ is open since the first projection $\mathbb{R} \times \Delta \xrightarrow{pr_1} \mathbb{R}$ is open, s is open, and $p = pr_1 \circ s$. The amenability of G comes from standard facts for groupoids (cf. [1]). ∎

The main result of the section is

Theorem 2.2. *Let M be a Hausdorff, separable manifold with a continuous action of $\mathbb{T}^n$ such that the quotient space Δ is a locally closed part of a Hausdorff manifold N endowed with a smooth action β of $\mathbb{R}^n$. If*

1. *the groupoid G is étale, amenable and the bundle map $G \longrightarrow \mathbb{R}$ is open (cf. Proposition 2.1),*
2. *the projection $M \xrightarrow{J} N$ is smooth and has a continuous section σ such that, for every $k \in \mathbb{Z}^n$, the functions $M_k \xrightarrow{k_\sigma} \mathbb{T}$ are smooth,*

then M admits a Poisson bracket $\{.,.\}_\Omega$ such that G is a deformation groupoid of (M, Ω).

This result is similar to that of Rieffel [21,22] but requires quite different technical conditions.

The proof of Theorem 2.2 consists in applying Proposition 0.5. Theorem 1.3 shows that condition 1 of Proposition 0.5 holds. In order to show that condition 2 holds, we define for any locally closed subset Y of a manifold $\widetilde{Y}$ the space $\mathrm{C}^\infty_\mathrm{c}(Y \subset \widetilde{Y})$ of continuous compactly supported functions $f \in \mathrm{C}_\mathrm{c}(Y)$ which admit a smooth extension $\widetilde{f} \in \mathrm{C}^\infty(\widetilde{Y})$. We have the following lemmas.

Lemma 2.3. *Under the assumptions of the Theorem 2.2, $\mathcal{Q} = \mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G})$ is a dense sub-*-algebra of $\mathrm{C}^*(G)$.*

Proof. The space $\mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G})$ is clearly a linear subspace of $\mathrm{C}_\mathrm{c}(G)$, and is stable by involution, since, for any f with extension $\widetilde{f} \in \mathrm{C}^\infty(\widetilde{G})$, the map f^* has an extension $g \longmapsto \overline{\widetilde{f}(g^{-1})}$ which is smooth, since the inverse map of $\widetilde{G}$ is a diffeomorphism.

Since the open bisections $G_k = G \cap (\mathbb{R} \times N \times \{k\})$ form a partition of G for $k \in \mathbb{Z}^n$, we obtain

$$\mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G}) = \bigoplus_{k \in \mathbb{Z}^n} \mathrm{C}^\infty_\mathrm{c}(G_k \subset \mathbb{R} \times N \times \{k\}).$$

Then, to prove the stability of $\mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G})$ by product, one has only to show that, for every $l, m \in \mathbb{Z}^n$ and every $f \in \mathrm{C}^\infty_\mathrm{c}(G_k \subset \mathbb{R} \times N \times \{l\})$ and $f' \in \mathrm{C}^\infty_\mathrm{c}(G_l \subset \mathbb{R} \times N \times \{m\})$, we have $f * f' \in \mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G})$. This follows easily from the formula

$$(f * f')(\hbar, y, k) = \begin{cases} f(\hbar, \beta_{\hbar l}(y), l) f'(\hbar, y, m) & \text{if } k = l + m \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{C}_\mathrm{c}(G)$ is dense in $\mathrm{C}^*(G)$, then $\mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G})$ is dense in $\mathrm{C}^*(G)$ if every $f \in \mathrm{C}_\mathrm{c}(G)$ can be approximated uniformly by functions $f_n \in \mathrm{C}^\infty_\mathrm{c}(G \subset \widetilde{G})$, since the topology induced by $\mathrm{C}^*(G)$ on $\mathrm{C}_\mathrm{c}(G)$ is the sup-norm topology. To prove such an approximation, we note that, since G is locally closed in $\widetilde{G}$, a function $f_n \in \mathrm{C_c}(G)$ is in $\mathrm{C}_c^\infty(G \subset \widetilde{G})$ if and only if it admits a smooth *and compactly supported* extension $\widetilde{f_n} \in \mathrm{C}_c^\infty(\widetilde{G})$. Any $f \in \mathrm{C_c}(G)$ is compactly supported, so it admits a continuous extension $\widetilde{f}$ on $\widetilde{G}$ such that Supp $\widetilde{f} \cap G$ is compact. This extension $\widetilde{f}$ can be uniformly approximated by $\widetilde{f_n} \in \mathrm{C}_c^\infty(\widetilde{G})$, and using a smooth partition of unity, we can suppose that Supp $\widetilde{f_n} \subset$ Supp $\widetilde{f}$. Then, the restriction of $\widetilde{f_n}$ to G, $f_n = \widetilde{f_n}|_G$, is continuous and its support is included in Supp $\widetilde{f} \cap G$; hence it is compact. We have $f_n \in \mathrm{C}_c^\infty(G \subset \widetilde{G})$ and

$$\sup_G |f - f_n| \le \sup_{\widetilde{G}} \left|\widetilde{f} - \widetilde{f_n}\right| \xrightarrow[n\to\infty]{} 0.$$

So $\mathrm{C}_c^\infty(G \subset \widetilde{G})$ is dense in $\mathrm{C}^*(G)$ by the above characterization. ■

As a corollary, $\mathcal{Q}_0 = \mathrm{C}_c^\infty(G_0 \subset \widetilde{G}_0)$ is dense in $\mathrm{C}^*(G_0)$ (condition 2(a)). And from the formula

$$(\mathcal{I}^{-1} f_0)(x) = \sum_{k \in \widehat{\mathbb{T}^n/\mathbb{T}^n_{J(x)}}} f_0(J(x), k) k_\sigma(x),$$

of Theorem 1.3 and the assumptions that J and the k_σ are smooth, we get $\mathcal{I}^{-1}\mathcal{Q}_0 \subset \mathrm{C}_0^\infty(M)$ (condition 2(b)). Condition 2(c$'$) (and thus Theorem 2.2) follows from the next lemma.

Lemma 2.4. *Under the assumptions of Theorem 2.2, for every $f, f' \in \mathrm{C}_c^\infty(G \subset \widetilde{G})$ and every $\tilde{e} \in \mathrm{C}_c^\infty(\widetilde{G})$ which extends $f * f' - f' * f$ on $\widetilde{G}$ (resp. on a neighborhood in $\widetilde{G}$ of a given point of G_0), the restriction $d = \tilde{d}\big|_G$ of $\tilde{d} \in \mathrm{C}^\infty(\tilde{G})$ defined by*

$$\widetilde{d}(\hbar, y, k) = \frac{1}{\imath} \int_0^1 \frac{\partial \widetilde{e}}{\partial \hbar}(\hbar t, y, k)\, \mathrm{d}t$$

verifies the following.

1. *d is a continuous extension of the section $\hbar \longmapsto \frac{1}{\imath\hbar}[f_\hbar, f'_\hbar]$ on G (resp. on a neighborhood in G of the point of G_0 considered);*
2. *d has the same support as $f * f' - f' * f$, and thus is compactly supported;*
3. *the value of d on G_0 (resp. at the point of G_0 considered) only depends on f_0 and f_0', and is given by*

$$\{f_0, f_0'\}_0(0, y, k) = \frac{1}{\imath} \frac{\partial \tilde{e}}{\partial \hbar}(0, y, k).$$

Proof. 1. The 0-order Taylor integral formula gives

$$\tilde{e}(\hbar, y, k) = \tilde{e}(0, y, k) + \imath\hbar\tilde{d}(\hbar, y, k).$$

For every $(\hbar, y, k)$ in G (resp. in the neighbourhood of G_0 considered), since $(0, y, \hbar)$ is in G_0, and since $\mathrm{C}^*(G_0)$ is commutative, we have

$$\tilde{e}(0, y, k) = (f * f' - f' * f)(0, y, k) = 0.$$

Hence, we obtain on G

$$f * f' - f' * f = \imath\hbar d,$$

i.e. d is a continuous extension of $\frac{f*f'-f'*f}{\imath\hbar}$.

2. Since the bundle map $G \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}$ is open, the set $G - G_0$ is dense in G; hence the continuous extension d is unique, and the previous relation $f * f' - f' * f = \imath\hbar d$ shows that $f * f' - f' * f$ and d have the same support.
3. Since the map $(f, f') \longmapsto f * f' - f' * f$ is bilinear, so is $(f, f') \longmapsto d|_{G_0}$. Then, it is sufficient to prove that $f_0 = 0$ implies $d|_{G_0} = 0$. In the same way as for $\tilde{e}$ (we had $\tilde{e}|_{G_0} = 0$) we get

$$\exists \delta f \in \mathrm{C}_c^\infty(G \subset \widetilde{G}), \quad f = \imath\hbar\delta f.$$

Then, $\frac{f*f'-f'*f}{\imath\hbar} = \delta f * f' - f' * \delta f$; this can be continuously extended by 0 at $\hbar = 0$. ■

3. Computation of the Poisson bracket

Adding to the hypothesis of Theorem 2.2 the assumption that the action α is smooth and some other technical assumptions, one can compute explicitly the Poisson structure Ω on M such that G is a deformation groupoid of (M, Ω). We introduce some notation and conventions. We identify $\mathbb{R}^n$ with its Lie algebra $Lie(\mathbb{R}^n)$. We fix a basis $E_1, \dots, E_n$ of $\mathbb{R}^n$, and we identify $\mathbb{R}^n$ with $Lie(\mathbb{T}^n)$, the Lie algebra of $\mathbb{T}^n$, and with its linear dual $Lie(\mathbb{T}^n)^*$; in particular $\mathbb{Z}^n = \widehat{\mathbb{T}^n}$ is viewed as a lattice in $\mathbb{R}^n$, and denoting by $(.\mid.)$ the duality bracket of $\mathbb{R}^n$, and by $Lie(\mathbb{T}^n) \xrightarrow{\exp} \mathbb{T}^n$ the exponential map, we get for every $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$

$$\left(E_i \mid E_j\right) = \delta_{i,j}, \quad (k \mid E_i) = k_i, \quad \langle k, \exp X\rangle = e^{\imath (k|X)}.$$

The vector field on M (resp. N) of the infinitesimal action of α (resp. β) in the direction $X \in \mathbb{R}^n$ will be denoted by ξ_X^α (resp. ξ_X^β), *i.e.* :

$$\forall x \in M, \quad \xi_X^\alpha(x) = \frac{\mathrm{d}}{\mathrm{d}\hbar}\alpha_{\exp(\hbar X)}(x)\Big|_{\hbar=0}, \quad \text{and} \quad \forall y \in N, \quad \xi_X^\beta(y) = \frac{\mathrm{d}}{\mathrm{d}\hbar}\beta_{\hbar X}(y)\Big|_{\hbar=0}.$$

We denote by $E_1^*, \dots, E_n^*$ the functions on G_0 defined by

$$E_i^*(0, y, k) = (k|E_i).$$

The first step is:

Proposition 3.1. *Under the assumptions of Theorem 2.2, let Ω be the Poisson bivector on M such that G is a deformation groupoid of (M, Ω). For any $f, f' \in C_c^\infty(G \subset \widetilde{G})$, and any extension $\widetilde{f}, \widetilde{f}' \in C_c^\infty(\widetilde{G})$ of f and f', we define $f_0 = f|_{G_0}$ and $f_0' = f'|_{G_0}$. If $\mathcal{I}$ is the isomorphism constructed in Theorem 1.3, we have*

$$\{\mathcal{I}^{-1} f_0, \mathcal{I}^{-1} f_0'\}_\Omega = \frac{1}{\imath}\sum_{i=1}^n \mathcal{I}^{-1}\left[d\widetilde{f}\left(\xi_{E_i}^\beta\right)\Big|_{G_0}\right]\left[\mathcal{I}^{-1}(E_i^* f_0')\right] - \left[\mathcal{I}^{-1}(E_i^* f_0)\right]\mathcal{I}^{-1}\left[d\widetilde{f}'\left(\xi_{E_i}^\beta\right)\Big|_{G_0}\right].$$

Proof. Since the Poisson bracket $\{.,.\}_\Omega$ on M is given by the relation

$$\mathcal{I}^{-1}\{f_0, f_0'\}_0 = \{\mathcal{I}^{-1} f_0, \mathcal{I}^{-1} f_0'\}_\Omega,$$

we have only to prove that

$$\{f_0, f_0'\}_0 = \frac{1}{\imath}\sum_{i=1}^n d\widetilde{f}\left(\xi_{E_i}^\beta\right)\Big|_{G_0} * (E_i^* f_0') - (E_i^* f_0) * d\widetilde{f}'\left(\xi_{E_i}^\beta\right)\Big|_{G_0}.$$

We can restrict to functions f, f' with extensions $\widetilde{f}, \widetilde{f}'$ both supported in $\mathbb{R} \times N \times \{l\}$ and $\mathbb{R} \times N \times \{m\}$, for some $l, m \in \mathbb{Z}^n$. Then, $f * f'$ and $f' * f$ are both supported in $\mathbb{R} \times N \times \{m + l\}$.

Since we assumed that G is étale, every element in $G_0 \cap (\mathbb{R} \times N \times \{m + l\})$ has a neighborhood V in G such that, for every $(\hbar, y, m + l)$ in V, the elements

$$(\hbar, y, l), (\hbar, \beta_{\hbar l}(y), m), (\hbar, y, m), (\hbar, \beta_{\hbar m}(y), l)$$

are all in G. On V, we get

$$f * f' - f' * f = \widetilde{e},$$

where $\widetilde{e} \in C^\infty(\widetilde{G})$ is the function defined by

$$\widetilde{e}(\hbar, y, m + l) = \widetilde{f}(\hbar, \beta_{\hbar l}(y), m)\widetilde{f}'(\hbar, y, l) - \widetilde{f}'(\hbar, \beta_{\hbar m}(y), l)\widetilde{f}(\hbar, y, m).$$

From Lemma 2.4, we get

$$\{f_0, f_0'\}_0(0, y, k) = \frac{1}{\imath}\frac{\partial \widetilde{e}}{\partial \hbar}(0, y, k).$$

We compute

$$\frac{\partial \widetilde{e}}{\partial \hbar}(\hbar, y, k) = \left(\frac{\partial \widetilde{f}}{\partial \hbar}(\hbar, \beta_{\hbar l}(y), m) + d\widetilde{f}\left(\frac{\mathrm{d}\beta_{\hbar l}}{\mathrm{d}\hbar}(y)\right)(\hbar, \beta_{\hbar l}(y), m)\right)\widetilde{f}'(\hbar, y, l)$$
$$+ \widetilde{f}(\hbar, \beta_{\hbar l}(y), m)\frac{\partial \widetilde{f}'}{\partial \hbar}(\hbar, y, l) - \widetilde{f}'(\hbar, \beta_{\hbar m}(y), l)\frac{\partial \widetilde{f}}{\partial \hbar}(\hbar, y, m)$$
$$- \left(\frac{\partial \widetilde{f}'}{\partial \hbar}(\hbar, \beta_{\hbar m}(y), l) + d\widetilde{f}'\left(\frac{\mathrm{d}\beta_{\hbar m}}{\mathrm{d}\hbar}(y)\right)(\hbar, \beta_{\hbar m}(y), l)\right)\widetilde{f}(\hbar, y, m).$$

For $\hbar = 0$, since $\beta_0(y) = y$, we obtain

$$\frac{\partial \widetilde{e}}{\partial \hbar}(0, y, k) = d\widetilde{f}(\xi_l^\beta)(0, y, m)\widetilde{f}'(0, y, l) - d\widetilde{f}'(\xi_m^\beta)(0, y, l)\widetilde{f}(0, y, m).$$

Since $\xi_l^\beta = \sum_{i=1}^n (l \mid E_i)\,\xi_{E_i}^\beta$ we get then

$$\{f_0, f_0'\}_0(0, y, k) = \frac{1}{\imath}\sum_{i=1}^n d\widetilde{f}(\xi_{E_i}^\beta)(0, y, m)E_i^*(0, y, l)\widetilde{f}'(0, y, l) - d\widetilde{f}'(\xi_{E_i}^\beta)(0, y, l)E_i^*(0, y, m)\widetilde{f}(0, y, m),$$

i.e.

$$\{f_0, f_0'\}_0 = \frac{1}{\imath}\sum_{i=1}^n d\widetilde{f}\left(\xi_{E_i}^\beta\right)\Big|_{G_0} * (E_i^* f_0') - (E_i^* f_0) * d\widetilde{f}'\left(\xi_{E_i}^\beta\right)\Big|_{G_0}. \quad \blacksquare$$

The second step is to compute $\mathcal{I}^{-1} d\widetilde{f}\left(\xi_{E_i}^\beta\right)\Big|_{G_0}$ and $\mathcal{I}^{-1}(E_i^* f_0)$ with respect to $\mathcal{I}^{-1} f_0$. We use some additional conditions to obtain:

Theorem 3.2. *Assume the hypothesis of Theorem 2.2, along with the following:*

1. *the action α of $\mathbb{T}^n$ on M is smooth;*
2. *the set $\Delta(\infty) = \{y \in \Delta \mid \mathbb{T}^n_y = \{1\}\}$ is open in N and dense in Δ*[3]*;*
3. *the restricted map $\Delta(\infty) \times \mathbb{T}^n \xrightarrow{\rho} M_\infty$ (cf. Lemma 1.4) is a diffeomorphism.*

Then, the Poisson structure of Theorem 2.2 is given on M_∞ by the bivector

$$\Omega = \sum_{i=1}^n \xi_{E_i}^\alpha \wedge \rho_*(\xi_{E_i}^\beta).$$

Proof. Remark that, for $\mathcal{Q} = \mathrm{C}_\mathrm{c}^\infty(G \subset \widetilde{G})$, we have both

$$\mathcal{Q}_0 = \mathrm{C}_\mathrm{c}^\infty(G_0 \subset \widetilde{G}_0) \quad \text{and} \quad \mathcal{I}^{-1}\mathcal{Q}_0 \subset \mathrm{C}_0^\infty(M),$$

and hence $\mathcal{I}^{-1}\mathcal{Q}_0$ is dense in $\mathrm{C}_0^\infty(M)$ since $\mathrm{C}_\mathrm{c}^\infty(G_0 \subset \widetilde{G}_0)$ is dense in $\mathrm{C}^*(G_0)$. So we have only to prove that, for every $f_0, f_0' \in \mathrm{C}_\mathrm{c}^\infty(G_0 \subset \widetilde{G}_0)$,

$$\{\mathcal{I}^{-1} f_0, \mathcal{I}^{-1} f_0'\}_\Omega = \left\langle d(\mathcal{I}^{-1} f_0) \otimes d(\mathcal{I}^{-1} f_0') \;\middle|\; \sum_{i=1}^n \xi_{E_i}^\alpha \wedge \rho_*(\xi_{E_i}^\beta)\right\rangle,$$

i.e.

$$\{\mathcal{I}^{-1} f_0, \mathcal{I}^{-1} f_0'\}_\Omega = \sum_{i=1}^n d\left(\mathcal{I}^{-1} f_0\right)\left(\xi_{E_i}^\alpha\right) d\left(\mathcal{I}^{-1} f_0'\right)\left(\rho^*\left(\xi_{E_i}^\beta\right)\right)$$
$$- d\left(\mathcal{I}^{-1} f_0'\right)\left(\xi_{E_i}^\alpha\right) d\left(\mathcal{I}^{-1} f_0\right)\left(\rho^*\left(\xi_{E_i}^\beta\right)\right). \quad (1)$$

[3] The density occurs for example when the action α is effective.

Moreover, it is enough to prove formula (1) on $M_\infty = J^{-1}(\Delta(\infty))$, since it is dense in M. Formula (1) is a consequence of Proposition 3.1 and both formulas

$$\mathcal{I}^{-1}(E_i^* f_0) = \frac{1}{\imath} d(\mathcal{I}^{-1} f_0)(\xi^\alpha_{E_i}) \tag{2}$$

$$\mathcal{I}^{-1} d\widetilde{f}\left(\xi^\beta_{E_i}\right)\Big|_{G_0} = d\left(\mathcal{I}^{-1} f_0\right)(\rho_*(\xi^\beta_{E_i})). \tag{3}$$

Let us prove formula (2). Recall from Theorem 1.3 that, for all $x \in M$,

$$\mathcal{I}^{-1}(f_0)(x) = \sum_{k\in\mathbb{Z}^n} f(0, J(x), k) k_\sigma(x).$$

We have, for all $x \in M$,

$$dk_\sigma(\xi^\alpha_{E_i})(x) = \frac{\mathrm{d}}{\mathrm{d}t} k_\sigma\left(\alpha_{\exp(tE_i)}(x)\right)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\langle k, \exp(tE_i)\rangle\Big|_{t=0} k_\sigma(x) = \imath\,(k \mid E_i)\, k_\sigma(x),$$

and since J is invariant with respect to the action α, for all $x \in M$, we get

$$\begin{aligned} d_x(\mathcal{I}^{-1} f_0)(\xi^\alpha_{E_i}) &= \sum_{k\in\mathbb{Z}^n} f(0, J(x), k) dk_\sigma(\xi^\alpha_{E_i})(x) \\ &= \imath \sum_{k\in\mathbb{Z}^n} f(0, J(x), k) E_i^*(0, J(x), k) k_\sigma(x) \\ &= \imath \mathcal{I}^{-1}(E_i^* f_0)(x). \end{aligned}$$

For formula (3), since $\xi^\beta_{E_i}$ is a vector field on $\Delta(\infty)$, using conditions 2 and 3 and the definition of $\mathcal{I}$, we get, for all $s \in \mathbb{T}^n$ and $y \in \Delta(\infty)$,

$$\begin{aligned} d_{\rho(s,y)}\left(\mathcal{I}^{-1} f_0\right)(\rho_*(\xi^\beta_{E_i})) &= d_{(s,y)}\left(\mathcal{I}^{-1} f_0 \circ \rho\right)(\xi^\beta_{E_i}) \\ &= \sum_{k\in\mathbb{Z}^n} d_y \widetilde{f}(\xi^\beta_{E_i})\,\langle k, s\rangle \\ &= \left(\left(\mathcal{I}^{-1} d\widetilde{f}(\xi^\beta_{E_i})\Big|_{G_0}\right)\circ\rho\right)(s, y) \end{aligned}$$

and hence $d\left(\mathcal{I}^{-1} f_0\right)(\rho_*(\xi^\beta_{E_i})) = \mathcal{I}^{-1}\left(d\widetilde{f}(\xi^\beta_{E_i})\Big|_{G_0}\right)$. ■

4. Application to toric manifolds

First we recall some facts concerning toric manifolds. We will use the notations and conventions of Sections 1 and 3.

Definition 4.1. The smooth action α of a Lie group $\mathtt{G}$ on a symplectic manifold (M, ω) is *Hamiltonian* when there exists a so-called *moment map* $M \xrightarrow{J} Lie(\mathtt{G})^*$ such that

$$\forall \mathtt{f} \in \mathrm{C}^\infty(M), \quad \forall X \in Lie(\mathtt{G}), \quad d\mathtt{f}\left(\xi^\alpha_X\right) = \{(J \mid X), \mathtt{f}\}_\omega.$$

When M is connected, J is unique up to a constant.

A *toric manifold* is a compact and connected symplectic manifold (M, ω) endowed with an effective Hamiltonian action α of $\mathbb{T}^n$ such that M has (real) dimension $2n$.

In the 1980's Atiyah [2], and Guillemin and Sternberg [12] proved that, for a toric manifold, the image of the moment map $J(M)$ is a convex polytope in $\mathbb{R}^n = Lie(\mathbb{T}^n)^*$. And Delzant [9] completed this result with the following:

- the map J is $\mathbb{T}^n$-invariant and the quotient map is a bijection $M/\mathbb{T}^n \simeq J(M)$, and hence an homeomorphism. From now on, we identify $\Delta = M/\mathbb{T}^n$ with the polytope $J(M)$ in $N = \mathbb{R}^n$.

- Every isotropy group $\mathbb{T}^n_y$ is connected and depends only on the open face of the polytope Δ containing y; more precisely there exists a parametrization of Δ such that

  $$\Delta = \{y \in \mathbb{R}^n \mid \forall j \in \{1, \dots, n_0\}, (y \mid X_j) \geq \lambda_j\}, X_j \in \mathbb{Z}^n, \quad \lambda_j \in \mathbb{R}$$

  and for which

  $$Lie(\mathbb{T}^n_y) = span\{X_j \mid (y \mid X_j) = \lambda_j\}.$$

  In particular $\Delta(\infty)$ is the topological interior of Δ in $\mathbb{R}^n$.
- Two toric manifolds having the same polytope Δ are diffeomorphic, in a $\mathbb{T}^n$-equivariant symplectic way. Thus, a toric manifold is totally characterized by its polytope.
- When Δ is the polytope of a toric manifold M, there is an explicit construction of M (with action and symplectic form ω) from Δ.

We need to recall briefly the main steps of the latter construction; we use the parametrization of Δ described above.

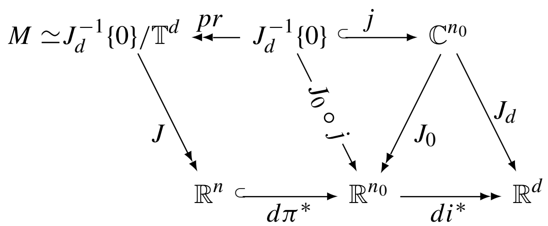
1. $\mathbb{T}^{n_0}$ has a (diagonal) Hamiltonian action on $\mathbb{C}^{n_0}$ endowed with its canonical symplectic structure $\omega_0 = \sum_{j=1}^{n_0} dx_j \wedge dy_j$. The moment map is given by

   $$J_0(z_1, \dots, z_{n_0}) = (\lambda_1, \dots, \lambda_{n_0}) + \frac{1}{2}\left(|z_1|^2, \dots, |z_{n_0}|^2\right) \in \mathbb{R}^{n_0}.$$

2. There exists an exact sequence of groups

   $$1 \longrightarrow \mathbb{T}^d \overset{i}{\hookrightarrow} \mathbb{T}^{n_0} \overset{\pi}{\twoheadrightarrow} \mathbb{T}^n \twoheadrightarrow 1, \quad (n_0 = n + d)$$

   such that $d\pi(F_j) = X_j$, where $F_1, \dots, F_{n_0}$ is the canonical basis of $\mathbb{R}^{n_0}$.
3. Using $Lie(\mathbb{T}^p)^* = \mathbb{R}^p$, let $\mathbb{R}^{n_0} \xrightarrow{di^*} \mathbb{R}^d$ be the cotangent map of i, and let us define $J_d = di^* \circ J_0$. Then, we have an inclusion map $J_d^{-1}\{0\} \overset{j}{\hookrightarrow} \mathbb{C}^{n_0}$ and $J_d^{-1}\{0\}$ is a subset of $\mathbb{C}^{n_0}$ invariant for the actions of $\mathbb{T}^{n_0}$ and of $\mathbb{T}^d$ – identified with its image by i in $\mathbb{T}^{n_0}$. Moreover, these two actions commute and we get a quotient action of $\mathbb{T}^n \simeq \mathbb{T}^{n_0}/\mathbb{T}^d$ on $J_d^{-1}\{0\}/\mathbb{T}^d$.
4. Using the Marsden–Weinstein symplectic reduction [15], the canonical projection $J_d^{-1}\{0\} \overset{pr}{\twoheadrightarrow} J_d^{-1}\{0\}/\mathbb{T}^d$ is a submersion, and hence $J_d^{-1}\{0\}/\mathbb{T}^d$ is a manifold of dimension $2n$ with symplectic structure ω such that $pr^*\omega = j^*\omega_0$. Moreover the action of $\mathbb{T}^n$ is Hamiltonian with moment map J. Calculations prove that the image of J is Δ, so we get the following commutative diagram:



$$J_0 \circ j = d\pi^* \circ J \circ pr$$

$$di^* \circ d\pi^* = 0$$

The main result of this section is:

Theorem 4.2. *Let (M, ω) be a toric manifold with the action α of $\mathbb{T}^n$, let $M \xrightarrow{J} N = \mathbb{R}^n$ be the moment map, and let β be the action of $\mathbb{R}^n$ on N by translation:*

$$\beta_X(y) = y + X.$$

Then, the groupoid G constructed in Theorem 2.2 is a deformation groupoid of (M, ω).

We will simply prove that the conditions of Theorem 2.2 are fulfilled, and then use Theorem 3.2 to prove that the Poisson structure Ω so obtained is the same as the one coming from ω, *i.e.* $\{., .\}_\omega = \{., .\}_\Omega$.

Condition 1 of Theorem 2.2 is fulfilled using Proposition 2.1 and the following lemma.

Lemma 4.3. *Under the assumptions of Theorem 4.2, for every $k \in \mathbb{Z}^n$, the following set is open in $\Delta \times \mathbb{R}^n$:*

$$s(G_k) = \{(\hbar, k) \in \mathbb{R} \times \Delta \mid y, y + \hbar k \in \Delta(k) \text{ and } \mathbb{T}^n_y = \mathbb{T}^n_{y+\hbar k}\}.$$

Proof. Recall that the isotropy subgroups $\mathbb{T}^n_y$ are connected, and that

$$Lie(\mathbb{T}^n_y) = span\{X_j \mid (y \mid X_j) = \lambda_j\}.$$

So we obtain

$$\Delta(k) = \{y \in \Delta \mid \forall j \in \{1, \dots, n_0\}, (y \mid X_j) = \lambda_j \Rightarrow (k \mid X_j) = 0\} = \Delta \cap U(k),$$

where $U(k)$ is the open subset of $\mathbb{R}^n$ defined by

$$U(k) = \{y \in \mathbb{R}^n \mid \forall j \in \{1, \dots, n_0\}, (k \mid X_j) \neq 0 \Rightarrow (y \mid X_j) > \lambda_j\}.$$

Since we have

$$\left(y + \hbar k \mid X_j\right) = \left(y \mid X_j\right) + \hbar \left(k \mid X_j\right),$$

Then, we get that

$$y \in \Delta(k) \Rightarrow Lie(\mathbb{T}^n_y) \subset Lie(\mathbb{T}^n_{y+\hbar k}).$$

In the same way, we get $y + \hbar k \in \Delta(k) \Rightarrow Lie(\mathbb{T}^n_{y+\hbar k}) \subset Lie(\mathbb{T}^n_y)$. Hence $y, y + \hbar k \in \Delta(k)$ implies that $\mathbb{T}^n_y = \mathbb{T}^n_{y+\hbar k}$ and we have furthermore

$$y, y + \hbar k \in \Delta(k) \Leftrightarrow y \in \Delta(k) \quad \text{and} \quad y + \hbar k \in U(k).$$

So

$$s(G_k) = \{(\hbar, k) \in \mathbb{R} \times \Delta \mid y \in \Delta(k) \text{ and } y + \hbar k \in U(k)\}$$

is open in $\mathbb{R} \times \Delta$. ■

Condition 2 of Theorem 2.2 is fulfilled using the two following lemmas. For the action of $\mathbb{T}^{n_0}$ in $\mathbb{C}^{n_0}$ we have $J_0(\mathbb{C}^{n_0}) = \lambda + (\mathbb{R}_+)^{n_0} = \Delta_0$, and there is a natural section σ_0 of J_0 given, for every $(w_1, \dots, w_{n_0}) \in \lambda + (\mathbb{R}_+)^{n_0}$, by

$$\sigma_0(w_1, \dots, w_{n_0}) = \left(\sqrt{2w_1 - \lambda_1}, \dots, \sqrt{2w_{n_0} - \lambda_{n_0}}\right).$$

Lemma 4.4. *For every toric manifold (M, ω) constructed as before, the map $\sigma = pr \circ \sigma_0 \circ d\pi^*$ is well defined and is a section of $M \xrightarrow{J} \Delta \subset \mathbb{R}^n$.*

Proof. Using the previous big commuting diagram and $J_0 \circ \sigma_0 = \text{id}$, we get for every $y \in \Delta$

$$J_d((\sigma_0 \circ d\pi^*)(y)) = (di^* \circ d\pi^*)(y) = 0,$$

and hence $(\sigma_0 \circ d\pi^*)(y)$ is in $J_d^{-1}\{0\}$, so σ is well defined.

The same kind of calculation proves

$$d\pi^* \circ (J \circ \sigma) = d\pi^*.$$

Since π is surjective, $d\pi^*$ is injective, and hence $J \circ \sigma = \text{id}$. ■

In particular, this section σ is continuous. Moreover, since the restriction of σ_0 to $\Delta_0(0) = \lambda + (\mathbb{R}^*_+)^{n_0}$ is smooth, we get that σ is smooth on $\Delta(\infty)$, and hence the map $\Delta(\infty) \times \mathbb{T}^n \xrightarrow{\rho} M$ is smooth.

Lemma 4.5. *For every toric manifold (M, ω) constructed as before and with respect to the previous σ, the map $M_k \xrightarrow{k_\sigma} \mathbb{T}$ is smooth for all $k \in \mathbb{Z}^n$.*

Proof. Considering the diagonal action of $\mathbb{T}^{n_0}$ on $\mathbb{C}^{n_0}$ with the section σ_0, in the same way as in Section 1 for the action α of $\mathbb{T}^n$ on M with the section σ, one can define:

- a map $\Delta_0 \times \mathbb{T}^{n_0} \xrightarrow{\rho_0} \mathbb{C}^{n_0}$;
- the sets $\Delta_0(l)$ and $\mathbb{C}^{n_0}_l = \rho_0(\Delta_0(l) \times \mathbb{T}^{n_0})$, for every $l = (l_1, \dots, l_{n_0}) \in \mathbb{Z}^{n_0}$;
- the map $\mathbb{C}^{n_0}_l \xrightarrow{l_{\sigma_0}} \mathbb{T}$.

Explicitly,

$$\mathbb{C}^{n_0}_l = \{(z_1, \ldots, z_{n_0}) \in \mathbb{C}^{n_0} \mid l_j \neq 0 \Rightarrow z_j \neq 0\}$$

and, since $\sigma_0(J_0(z_1, \ldots, z_{n_0})) = (|z_1|, \ldots, |z_{n_0}|)$, we have

$$l_{\sigma_0}(z_1, \ldots, z_{n_0}) = \prod_{j|l_j \neq 0} \left(\frac{z_j}{|z_j|}\right)^{l_j},$$

and hence l_{σ_0} is smooth on $\mathbb{C}^{n_0}_l$.

For $k \in \mathbb{Z}^n$, we apply this construction to the particular case $l = d\pi^*(k) \in \mathbb{Z}^{n_0}$. Then, one can check that the following diagram is commutative:

$$\begin{array}{ccccc} M_k & \xleftarrow{pr} & pr^{-1}(M_k) & \xhookrightarrow{j} & \mathbb{C}^{n_0}_l \\ & {\scriptstyle k_\sigma}\searrow & & \swarrow{\scriptstyle l_{\sigma_0}} & \\ & & \mathbb{T} & & \end{array} \qquad l_{\sigma_0} \circ j = k_\sigma \circ pr$$

Since pr is a submersion and $l_{\sigma_0} \circ j$ is smooth, k_σ is smooth too. ■

So the conditions of Theorem 2.2 are satisfied. To use Theorem 3.2 it remains only to prove that the map $\Delta(\infty) \times \mathbb{T}^n \xrightarrow{\rho} M_\infty$ is a diffeomorphism; we already know that ρ is one-to-one and smooth. We obtain that ρ is a diffeomorphism since it pulls back a nondegenerate 2-form onto a nondegenerate 2-form, as is established in the following lemma.

Lemma 4.6. *For any toric manifold (M, ω) constructed as before, and for any $(y, s) \in \Delta(\infty) \times \mathbb{T}^n$, identifying the tangent bundle of $\Delta(\infty) \times \mathbb{T}^n$ at (y, s) with $\mathbb{R}^n \times \mathbb{R}^n$, we get*

$$\forall (Y, X), (Y', X') \in \mathbb{R}^n \times \mathbb{R}^n, \quad \rho^*\omega((Y, X), (Y', X')) = \left(X \mid Y'\right) - \left(Y \mid X'\right).$$

Proof. For any $y \in \Delta(\infty)$ and $s \in \mathbb{T}^n$, we get

$$d_{y,s}\rho(Y, X) = \xi^\alpha_X(\rho(y, s)) + d_{\sigma(y)}\alpha_s(d_y\sigma(Y)).$$

So, using the relation between ω and J given by $\omega\left(\xi^\alpha_X, T\right) = d\,(J \mid X)\,(T)$, we get

$$\rho^*\omega((0, X), (0, X')) = \omega\left(\xi^\alpha_X, \xi^\alpha_{X'}\right) = 0.$$
$$\rho^*\omega((0, X), (Y, 0)) = \omega\left(\xi^\alpha_X, d\alpha_s(d\sigma(Y))\right) = (X \mid Y).$$

Then, since $\sigma_0^*\omega_0 = 0$, we get, using $pr^*\omega = j^*\omega_0$, that

$$\sigma^*\omega = (pr \circ \sigma_0 \circ d\pi^*)^*\omega = (\sigma_0 \circ d\pi^*)^*(pr^*\omega) = (d\pi^*)^*(\sigma_0^*\omega_0) = 0.$$

Hence,

$$\rho^*\omega((Y, 0), (Y', 0)) = (\alpha_s \circ \sigma)^*\omega((Y, 0), (Y', 0)) = \sigma^*\omega((Y, 0), (Y', 0)) = 0. \quad ■$$

Moreover, using the fact that $\xi^\beta_Y = Y$ since β acts by translations, we get

$$d_{\sigma(y)}\alpha_s\left(d_y\sigma(Y)\right) = \rho^*\left(\xi^\beta_Y\right).$$

Hence, it follows immediately that the Poisson bivector associated with ω is equal to the bivector Ω given in Theorem 3.2. So Theorem 4.2 is proved.

Finally, let us add a result on the structure of the C*-algebras $C^*(G_\hbar)$ which occur in the deformation:

Proposition 4.7. *For any $\hbar \in \mathbb{R}$, every irreducible representation of $C^*(G_\hbar)$ is finite dimensional. In particular $C^*(G_\hbar)$ is a C*-algebra of type I.*

Proof. For $\hbar = 0$, $C^*(G_0)$ is commutative, and hence all representations have dimension 1.

For $\hbar \neq 0$, we note that the action of $G_\hbar$ on its base $G_\hbar^{(0)}$ has no isotropy, i.e. $G_\hbar$ is a so-called principal groupoid. Moreover, every orbit of this action is finite, and hence closed. But every irreducible representation of the C^*-algebra of a principal groupoid is supported by a closed orbit. ∎

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